Hypergraph Convolutional Networks via Equivalency between Hypergraphs and Undirected Graphs

Zepeng Zhang

This paper [1] investigated the equivalency between generalized hypergraphs and undirected graphs, in the sense that a natural random walk on the hypergraph is equivalent to the natural random walk on a weighted undirected clique graph. A generalized hypergraph is defined as $\mathcal{H}(\mathcal{V}, \mathcal{E}, \mathbf{W}, \mathbf{Q}_1, \mathbf{Q}_2)$, where $\mathbf{W} \in \mathbb{R}^{|\mathcal{E}| \times |\mathcal{E}|}$ is the edge-weights diagonal matrix with entries $W(e, e) = \omega(e)$, $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are in-edge vertex weight matrix and the out-edge vertex weight matrix, each of which can be edge-independent or edge-dependent. The clique graph of $\mathcal{H}(\mathcal{V}, \mathcal{E}, \mathbf{W}, \mathbf{Q}_1, \mathbf{Q}_2)$ is denoted as $\mathcal{G}^C$, which is an unweighted graph with vertices $\mathcal{V}$ and edge set $\mathcal{E}_{\mathcal{G}^C} = \{(u, v) : u, v \in e, e \in \mathcal{E}\}$. Actually, $\mathcal{G}^C$ turns all hyperedges into cliques.

Definition 1 (Unified random walk on hypergraphs)**.** The unified random walk on a hypergraph $\mathcal{H}$ is defined in a two-step manner: Given the current vertex u, first choose an arbitrary hyperedge e incident to u, with the probability

$$p_1 = \frac{w(e) \sum_{v \in \mathcal{V}} Q_2(v, e) \rho\left(\sum_{v \in \mathcal{V}} Q_2(v, e)\right) Q_1(u, e)}{\sum_{e \in \mathcal{E}} w(e) \sum_{v \in \mathcal{V}} Q_2(v, e) \rho\left(\sum_{v \in \mathcal{V}} Q_2(v, e)\right) Q_1(u, e)},$$

then choose an arbitrary vertex v from e, with the probability

$$p_2 = \frac{Q_2(v, e)}{\sum_{v \in \mathcal{V}} Q_2(v, e)},$$

where $\omega(e)$ denotes the hyperedge weight and $\rho(\cdot)$ is a real-valued function that acts on the degree of the hyperedge and is used to control the random process. Consider an example with $\rho = (\cdot)^\sigma$. For positive values of σ, the hyperedges with large degree will dominate the random process. Conversely, when σ is negative, hyperedges with small degree are likely to drive the random walk process.

Theorem 1 (Equivalency between generalized hypergraph and weighted undigraph)**.** *When $\mathcal{H}(\mathcal{V}, \mathcal{E}, \mathbf{W}, \mathbf{Q}_1, \mathbf{Q}_2)$ satisfies any of the condition bellow:*

Condition (1): $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are both edge-independent;

Condition (2): $\mathbf{Q}_1 = k\mathbf{Q}_2$ ($k \in \mathbb{R}$),

there exists a weighted undirected clique graph $\mathcal{G}^C$ such that a random walk on $\mathcal{H}$ is equivalent to a random walk on $\mathcal{G}^C$.

Corollary 1. *Let $\hat{\mathbf{D}}_v$ be a $|\mathcal{V}| \times |\mathcal{V}|$ diagonal matrix with entries $\hat{D}_v(v, v) := \hat{d}(v) := \sum_{e \in \mathcal{E}} w(e)\delta(e)\rho(\delta(e))Q_2(v, e)$. No matter $\mathcal{H}$ satisfies condition (1) or condition (2) in the above theorem, it obtains the unified explicit of stationary distribution π and Laplacian matrix $\mathbf{L}$ as*

$$\pi = \frac{\mathbf{1}^\top \hat{\mathbf{D}}_v}{\mathbf{1}^\top \hat{\mathbf{D}}_v \mathbf{1}} \text{ and } \mathbf{L} = \mathbf{I} - \hat{\mathbf{D}}_v^{-1/2} \mathbf{Q}_2 \mathbf{W} \rho\left(\mathbf{D}_e\right) \mathbf{Q}_2^\top \hat{\mathbf{D}}_v^{-1/2}.$$

Based on the above results, graph convolutions on hypergraphs are transformed to graph convolutions on the corresponding undirected graphs. Then the GCNNs on hypergraphs can be constructed by replacing the Laplacian in normal GCNNs to be the one corresponds to hypergraphs.

References

[1] Jiying Zhang, Fuyang Li, Xi Xiao, Tingyang Xu, Yu Rong, Junzhou Huang, and Yatao Bian. Hypergraph convolutional networks via equivalency between hypergraphs and undirected graphs. *arXiv preprint arXiv:2203.16939*, 2022. (document)

Preprint. Under review.